

UDC 512.542

Recognizability by spectrum of alternating groupsGorshkov I.B.¹*Dedicated to 70-th anniversary of V. D. Mazurov*

Review: The spectrum of a group is the set of its element orders. A finite group G is said to be recognizable by spectrum if every finite group that has the same spectrum as G is isomorphic to G . We prove that the simple alternating groups A_n are recognizable by spectrum when $n \neq 6, 10$. This implies that every finite group with the same spectrum as that of a finite nonabelian simple group, has at most one nonabelian composition factor.

Key words: finite group, simple group, alternating group, spectrum of a group, recognizability by spectrum.

Suppose G is a finite group, $\pi(G)$ is the set of prime divisors of its order, $\omega(G)$ is the spectrum of G , i.e. the set of its element orders. The Gruenberg-Kegel graph, or the prime graph, $GK(G)$ is defined as follows. The vertex set of the graph is $\pi(G)$. Two distinct primes p and q of $\pi(G)$ seen as vertices of the graph $GK(G)$, are connected by an edge if, and only if, $pq \in \omega(G)$. A group G is said to be recognizable by spectrum (shortly, recognizable) if for every finite group L the equality $\omega(L) = \omega(G)$ implies that $L \simeq G$. In [1], [2] the following groups were proved to be recognizable: A_n , $n \geq 5$, $n \notin \{6, 10\}$ and $n - p \in \{0, 1, 2\}$, where p is a maximal prime number which does not exceed n . The proof is based on the fact that the graph $GK(A_n)$ is disconnected, and the prime p forms its connected component, which is not true in general case. The group A_6 was proved to be irreducible in [3]. In [4] it was shown that A_{10} is irreducible. The groups A_{16} and A_{22} were proved to be recognizable in [2], [5] respectively. In particular, the question on recognizability is settled for all alternating groups A_k , where $5 \leq k \leq 25$.

A finite simple group L is said to be quasirecognizable, if every finite group G with $\omega(L) = \omega(G)$ has the unique nonabelian composition factor, which is isomorphic to L . It was shown in [6] that quasirecognizability of a finite simple alternating group implies its recognizability. In [7] a theorem describing the properties of chief series of groups with the same spectrum as that of the alternating group was proved. Using the above results we were able to prove that all simple alternating groups except for A_6 and A_{10} , are recognizable.

Theorem 1. *Suppose G is a finite group with $\omega(G) = \omega(A_n)$, where $n \geq 5$, $n \neq 6, 10$. Then G is isomorphic to A_n .*

This theorem gives an affirmative answer to Question 16.107 in the Kourovka Notebook. Also together with available results on groups A_6 and A_{10} it implies that the Question 16.27 is answered in the affirmative. Moreover, as it is indicated in the comment to Question 16.27, Theorem 1 and [8, Corollary 7.3] entail the following statement.

Theorem 2. *Suppose L is a finite nonabelian simple group and G is a finite group with $\omega(G) = \omega(L)$. Then G has at most one nonabelian composition factor. Moreover, if the group L is distinct from the groups $\text{PSL}_3(3)$, $\text{PSU}_3(3)$, $\text{PSp}_4(3)$, then G has exactly one nonabelian composition factor.*

¹Supported by Russian Foundation for Basic Research (Grants 11-01-00456, 12-01-31221, 12-01-90006).

Note that for every exception in Theorem 2 there exists a soluble group with the same spectrum.

§1. Preliminaries

Lemma 1. *Suppose K is a normal subgroup of a finite group G , $\overline{G} = G/K$, \overline{x} is the image of an element x of G in the group \overline{G} . If $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

Proof. See [9, Theorem 1.6.2].

Lemma 2. *Assume that a finite group G has a normal series $1 < K < M < G$, and pairwise distinct prime numbers p , q and r are such that p divides $|K|$, q divides $|M/K|$ and r divides $|G/M|$. Then the numbers p , q and r can not be pairwise non-adjacent in $GK(G)$.*

Proof. See [10, Lemma 1.1].

Lemma 3. *Every odd number of $\pi(\text{Out}(P))$, where P is a finite simple group, belongs to the spectrum of the group P or does not exceed $m/2$, where $m = \max_{p \in \pi(P)} p$.*

Proof. See [7, Lemma 14].

Lemma 4. *If $n \geq 5$, $n \neq 6, 10$ and $n = r + m$, where r is a prime which is greater than 3, $m \in \{0, 1, 2\}$, then the group A_n is recognizable.*

Proof. See [1] and [2].

Lemma 5. *If $n > 6$, then $\omega(A_n) \not\subseteq \omega(S_{n-1})$.*

Proof. See [2, Lemma 1.2]

Lemma 6. *If $n > 6$, then $\omega(A_n) \neq \omega(S_n)$.*

Proof. See [2]

Lemma 7. *Suppose that a quotient group $H = T/N$ of a finite group T is isomorphic to a symmetric or alternating group of degree m , where $m \geq 5$ and $N \neq 1$. Then $\omega(T/N) \neq \omega(T)$.*

Proof. See [6].

Lemma 8. *Suppose that $\omega(G) = \omega(A_n)$, $n > 21$. Then a chief series of G has a factor, which is isomorphic to A_k , for some k of $[p, n]$, where p is the maximal prime not exceeding n . In addition, no other factor contains p in its spectrum.*

Proof. See [7].

Lemma 9. *Suppose that $g \in A_n$ is an element of order r , where r is a prime and $n/2 < r < n - 2$, $n \geq 5$. Then $C_{A_n}(g) \simeq \langle g \rangle \times A_{n-r}$.*

Proof. An easy check.

§2. Proof of Theorem 1

Suppose that $n \geq 26$ is the least number such that $L = A_n$ is not recognizable, G is a finite group with $\omega(G) = \omega(L) = \omega$ and $G \not\cong L$, $1 = G_0 < G_1 < \dots < G_t = G$ is a chief series of G , $R_i = G_i/G_{i-1}$, $i \in [1, t]$. According to Lemma 8 there exists m such

that $R = R_m \simeq A_k$, $k \in [p, n]$, where p is the largest prime not exceeding n . We chose a chief series so that the quotient G/G_{m-1} has the least order. Then the group G/G_{m-1} possesses the unique minimal normal subgroup, which is isomorphic to A_k . So G/G_{m-1} is isomorphic either to A_k , or to S_k . By Lemma 4 the number $n - p$ is greater than 2. Put $\Pi = \{r | n/2 < r < p\}$, $\bar{\Pi} = \Pi \setminus \pi(G_{m-1})$. Note that numbers of Π are pairwise non-adjacent in $GK(L)$, and $\bar{\Pi}$ consists of those elements from Π , who divide just the order of the factor R . Let $\mu = \mu(G)$ denote the set of maximal under divisibility elements of $\omega(G)$. Note that $\omega(G)$ is uniquely determined by the set $\mu(G)$.

Lemma 10. *Under $n \neq 27$ there exist two distinct primes $r_1, r_2 \in \Pi$ such that $n - r_i \notin \{3, 4, 6, 10\}$, $i = 1, 2$.*

Proof. Since $p > r$ for every $r \in \Pi$ and $n - p > 2$, we have $n - r > 4$. Having $n \geq 42$, it is easy to check that $|\Pi| \geq 4$ (see, for example, [1, Lemma 1]). If $26 \leq n < 48$, $n \neq 27$, then the statement of the lemma can be checked directly. Under $n = 27$ the statement is false.

Lemma 11. *Let $y \in G$ be an element of order r , where $r \in \Pi \cup \{p\}$. If a Sylow r -subgroup of G is cyclic of prime order, then $\omega(C_G(y)) = \omega(C_L(x))$, where x is an r -element of the group L .*

Proof. If an element $z \in G$ centralizes some r -element of G , then there exists its conjugate z' , which centralizes y . Therefore, if $rm \in \omega$, then $rm \in \omega(C_G(y))$. The same we can say of the group L and the element x . Thus the sets $\mu(C_G(y))$ and $\mu(C_L(x))$ consist precisely of those elements of the set μ , who are divided by r .

Lemma 12. *Suppose that $y \in G$ is an element of order r , where $r \in \Pi \cup \{p\}$, and a Sylow r -subgroup of G is cyclic and has a prime order. Then $C_G(y) = \langle y \rangle \times M$, where $\mu(M) = \mu(Alt_{n-r})$.*

Proof. Put $M = C_G(y)/\langle y \rangle$. A Sylow r -subgroup of G is cyclic and has prime order. Hence $\langle y \rangle.M \simeq \langle y \rangle \times M$ and $\mu(M) = \{l/r | l \in \mu(C_G(y))\}$. If x is an r -element of L , then according to Lemma 9 we have $C_L(x) = \langle x \rangle \times N$, where $N \simeq Alt_{n-r}$. Therefore, $\mu(M) = \mu(N) = \mu(Alt_{n-r})$.

Lemma 13. $|\bar{\Pi}| \geq |\Pi| - 1$

Proof. Assume that there exists i such that $1 \leq i < m$ and $|R_i|$ is divided by two numbers of Π . Suppose that $R_i = T_1 \times T_2 \times \dots \times T_h$, where $T_j \simeq T_1$, $1 < j \leq h$, T_1 is a finite simple group. The fact that for every $r_1, r_2 \in \Pi$, $r_1 \neq r_2$, the set ω does not contain a number $r_1 r_2$ implies that $R_i = T_1$. By Lemma 8 number p does not divide $|R_i|$. Thus G/G_{i-1} possesses an element $g \notin R_i$ of order p , which acts by conjugation on the group R_i . According to Lemma 3 we have $R_i < C_{G/G_{i-1}}(g)$, but $\pi(C_{G/G_{i-1}}(g))$ has no numbers from Π . Hence for every $1 \leq i < m$ the intersection $\pi(R_i) \cap \Pi$ contains at most one number. Suppose that there exist two numbers $1 \leq h < l < m$ such that $|R_h|$ and $|R_l|$ are divided by distinct numbers from Π . The group G contains a normal series $1 < G_h < G_l < G_m$ and numbers $r, q, p \in \Pi$ such that r divides $|G_h|$, q divides $|G_l/G_h|$ and p divides $|G_m/G_l|$. The numbers r, q, p are pairwise non-adjacent in $GK(G)$ which contradicts Lemma 2. The lemma is proved.

Let us complete the proof of Theorem 1. Lemmas 10 and 13 imply that for every $n \geq 26$, $n \neq 27$, there exists $r \in \bar{\Pi}$ such that $n - r \notin \{3, 4, 6, 10\}$. We show that under $n = 27$ this statement is also true. Since $\Pi = \{17, 19\}$, we have to prove that $19 \in \bar{\Pi}$. Suppose that 19

divides $|G_{m-1}|$. We have $R \simeq A_k$, where $k \in [23, 27]$. The group R contains the Frobenius group F with the kernel of order 23 and the complement of order 11. By Lemma 2 the full preimage of the group F contains an element of order $11 \cdot 19$ or $23 \cdot 19$; contradiction.

Since a Sylow r -subgroup of the group R has order r and r does not divide $|G_{m-1}| \cdot |G/G_m|$, then a Sylow r -subgroup of the group G has order r also. Suppose that $g \in G$ is an element of order r , \bar{g} is the image of g in R . According to Lemma 12 we have $C_G(g) = \langle g \rangle \times M$, $\mu(M) = \mu(A_{n-r})$. The group A_s is recognizable for every $5 \leq s < n$, $s \neq 6, 10$ which implies that $M \simeq A_{n-r}$. Suppose that $C_i = C_G(g) \cap G_i$, $0 \leq i \leq t$. Since $n - r \geq 5$, the group M is simple, hence there exists i such that $M \leq C_i$ and $M \cap C_{i-1} = 1$. By Lemma 1 we have $C_R(\bar{g}) \simeq C_G(g)/(G_{m-1} \cap C_G(g))$. Since $C_R(\bar{g}) \neq \langle \bar{g} \rangle$, then $i = m$. By Lemma 9 we have $k = n$. Lemmas 6 and 7 imply that $G \simeq L$. Theorem 1 is proved.

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